Int. J. Heat Mass Transfer. Vol. 24, No. 5, pp. 887–890, 1981
Printed in Great Britain

0017-9310/81/050887-04 $02.00/0
Pergamon Press Ltd

DISPERSION EFFECTS ON THERMAL CONVECTION IN A HELE–SHAW CELL

ODDMUND KVERNVOLD* and PEDER A. TYVAND
Department of Mechanics, University of Oslo, Blindern, Oslo, Norway

(*Received* 24 *June* 1980)

Abstract—The influence of hydrodynamic dispersion on thermal convection in a vertical Hele–Shaw cell is studied theoretically. The supercritical, steady roll motion, the heat transport and the stability of the motion are investigated. The dispersion effects are found to be small, but qualitatively different from the corresponding effects in porous convection.

NOMENCLATURE

$\mathscr{D}$,	dimensionless dispersion tensor;
d,	channel width of Hele–Shaw cell;
Δh,	displacement thickness of boundary layer;
∇,	$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial z}$;
∇^2,	$\nabla \cdot \nabla$;
$-G$,	pressure gradient;
g,	acceleration of gravity;
h,	height of Hele–Shaw cell;
$\mathbf{i}, \mathbf{k}$,	unit vectors in x- and z-direction;
Nu,	Nusselt number;
p,	dimensionless pressure;
Ra,	Rayleigh number $d^2\gamma gh\Delta T/12\kappa\nu$;
T,	dimensionless temperature;
ΔT,	temperature difference between lower and upper boundary;
t,	dimensionless time;
$\mathbf{v}$,	velocity $u\mathbf{i} + w\mathbf{k}$;
x, y, z,	cartesian coordinates.

Greek letters

α,	dimensionless wave number;
γ,	coefficient of volume expansion;
θ,	dimensionless temperature;
κ,	thermal diffusivity;
ν,	kinematic viscosity;
μ,	dynamic viscosity.

INTRODUCTION

IN A RECENT paper Kvernvold and Tyvand [1] studied the effects of hydrodynamic dispersion on free convection in porous media. The steady two-dimensional motion, the heat transport and the stability of the motion were investigated. Interesting dispersion effects were found, strongly dependent on the coarseness of the porous material and on the Rayleigh number. New approximations to experimental data were obtained.

In the present paper the corresponding problem for a Hele–Shaw cell is studied. Here the lateral dispersion is zero and the motion is two-dimensional. These two restrictions lead to qualitatively new dispersion effects compared with porous convection. The present results extend the existing theory of convection in a Hele–Shaw cell [2–5].

* Present address: Det norske Veritas, 1322 Høvik, Norway.

FORMULATION OF THE PROBLEM

A Hele–Shaw cell is defined by two vertical walls of infinite horizontal extent with height h. The walls insulate for heat and are separated by a distance d. The space between the walls is occupied by a Newtonian fluid and bounded by two impermeable and perfectly heat-conducting planes at $z = 0$ and $z = h$. The x- and y-axes are directed along and normal to the walls, respectively. The walls are given by $y = \pm d/2$. The cell is unbounded in x-direction.

Dimensionless quantities are introduced by taking h, κ/h, ΔT, $hd^2/12\kappa\nu$, h^2/κ, as units of length, velocity, temperature, pressure and time, respectively. The governing equations may be written

$$\mathbf{v} + \nabla p - RaT\mathbf{k} = 0 \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla \cdot (\mathscr{D} \cdot \nabla T) \quad (3)$$

applying the Boussinesq and Hele–Shaw approximation [4, 5] and taking hydrodynamic dispersion [6] into account. Ra is the Hele–Shaw Rayleigh number

$$Ra = \frac{\mathrm{d}^2\gamma g\Delta Th}{12\kappa\nu}. \quad (4)$$

Wooding [2] calculated the dispersion in a Hele–Shaw cell with insulating walls, assuming a parabolic velocity distribution. His result can be put into the tensor form

$$\mathscr{D} = \mathbf{ii} + \mathbf{kk} + \frac{1}{210}\frac{d^2}{h^2}\mathbf{vv}. \qquad (5)$$

The dimensionless dispersion tensor (5) is valid for all Péclet numbers.

The governing equations (1)–(3) and (5) are included in the corresponding equations for a porous medium [1] by putting ε_1 equal to $d^2/(210h^2)$, ε_2 equal to zero and the y-dependence left out. The method of solution is the same as reported by Kvernvold and Tyvand [1]. Also the same boundary conditions are taken

$$w = \theta = 0 \quad \text{at} \quad z = 0, 1 \qquad (6)$$

which means impermeable, perfectly conducting boundaries. Here θ is defined by

$$T = (T)_{z=0} - z + \theta. \qquad (7)$$

RESULTS

Dispersion does not influence the onset of convection, neither by linear theory nor energy theory [7]. The critical Rayleigh number is

$$Ra_c = 4\pi^2 \qquad (8)$$

with a corresponding wave number $\alpha_c = \pi^2$.

The supercritical heat transport is given by the Nusselt number defined by

$$Nu = -\mathbf{k}\cdot\overline{(\mathscr{D}\cdot\nabla T)}_{z=0} = 1 - \overline{\left(\frac{\partial\theta}{\partial z}\right)}_{z=0} \qquad (9)$$

where the overbar denotes horizontal average. Here one basic difference from porous convection [1] is obvious. Because the lateral dispersion is zero, there are no terms with explicit dispersion dependence in the definition of the Nusselt number. In porous convection such terms are responsible for a rapid growth of the Nusselt number with the Rayleigh number.

The results for the Nusselt number are given in Tables 1 and 2 and Fig. 1. For given α and Ra, Nu is approximately a linear function of d^2/h^2. All the present calculations show that dispersion reduces the heat transport in a Hele–Shaw cell. Table 2 and Fig. 1 show Nu as a function of Ra/Ra_c for the wave number which gives maximum heat transport. This wave number is reduced by dispersion. A similar but much stronger reduction is present in porous convection [1].

In Fig. 1 the dispersion effect on the heat transport is compared with the porous medium case. The shaded area contains results for the Nusselt number in an isotropic porous medium with dispersion factor $D < 1/60$ [1]. In a Hele–Shaw cell Nu is smoothly reduced due to dispersion. For porous media, however, dispersion causes an increase in the heat transport which is strongly dependent on Ra, except in the range $Ra/Ra_c < 1.65$ where a small reduction is present.

The stability of the steady motion with respect to small disturbances has been investigated. In the Hele–Shaw approximation only Eckhaus disturbances (parallel-rolls) are possible. The results are shown in Fig. 2. The relatively large value $d^2/h^2 = 1/10$ is chosen for the purpose of illustration. It is possibly outside the range of validity for the Hele–Shaw approximation.

Table 1. Nusselt number at wave number $\alpha = \pi$

	d^2/h^2			
Ra/Ra_c	0	1/50	1/20	1/10
2	2.251	2.247	2.240	2.230
3	2.927	2.920	2.909	2.891
4	3.405	3.396	3.381	3.358
6	4.070	4.060	4.045	4.017

Table 2. Maximum Nusselt number and corresponding wave number

	$d^2/h^2 = 0$		$d^2/h^2 = 1/10$	
Ra/Ra_c	Nu	α/π	Nu	α/π
2	2.262	1.09	2.240	1.09
4	3.527	1.49	3.463	1.44
6	4.443	2.13	4.292	1.99
8	5.242	2.59	4.971	2.42
10	5.910	2.93	5.517	2.71
12	6.458	3.22	5.954	2.95

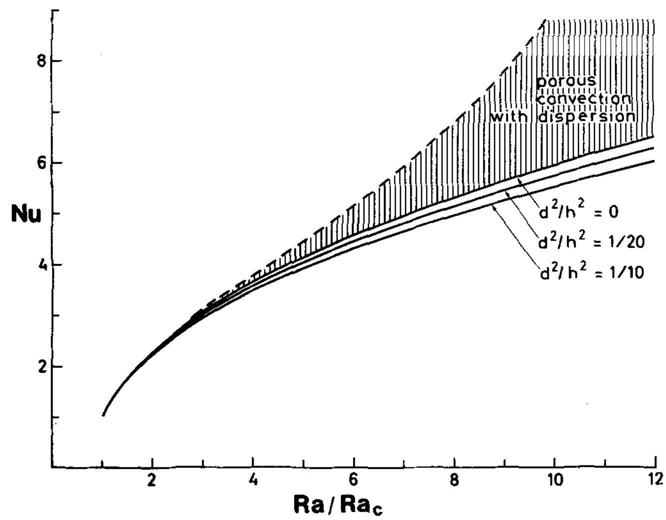

FIG. 1. Nu vs Ra/Ra_c for some values of d^2/h^2. Shaded area corresponds to free convection with dispersion in a porous medium, taken from Kvernvold and Tyvand [1]. We consider dispersion factor D not larger than 1/60, a case which is shown by the broken curve.

In Fig. 2 the case $d^2/h^2 = 0$ is also included [5]. The lower left-hand branch is corrected according to present calculations, being more accurate: up to $Ra/Ra_c = 3.8$ the most unstable disturbances are found to have exponential time dependence. Only in the range $3.8 < Ra/Ra_c < 7.8$ an oscillatory mode is present.

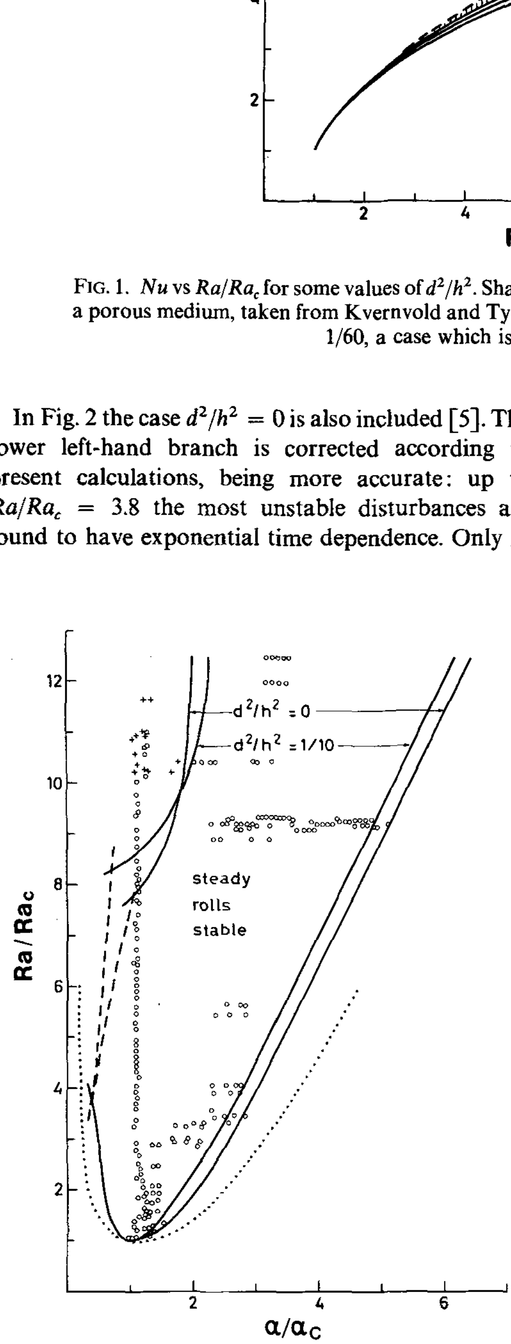

FIG. 2. Stability domains for steady roll motion in the α/α_c, Ra/Ra_c plane: —— exponential Eckhaus instability, – – – oscillatory Eckhaus instability. Experimental data by Koster and Müller [8]: Stable points in the α/α_c, Ra/Ra_c plane. ○ non-oscillatory; + oscillatory.

The stability domain in the α, Ra plane shows no tendency to close in our range of computation. At the right-hand branch dispersion reduces the upper limit of stable wave numbers. This reduction is remarkably constant for all $Ra/Ra_c > 2$. At the left hand branch dispersion is negligible for $Ra/Ra_c < 3.8$ where the most unstable mode of disturbance is exponential. When $3.8 < Ra/Ra_c < 9.7$, the lower limit of stable wave number is reduced due to dispersion, whereas it is increased for larger Rayleigh numbers. When $d^2/h^2 = 1/10$ the most unstable mode of disturbance is oscillatory in the range $3.8 < Ra/Ra_c < 8.3$.

Figure 2 also includes some experimental data obtained by Koster and Müller [8]. The experiments were made in a Hele–Shaw cell with $d/h = 1/41$. Then the effects of dispersion are negligible. The experiments confirm the right hand branch of the stability curve. Oscillatory behaviour is found at the left hand branch. However, it is not located where the theory predicts. Some of the results at the left-hand branch are in the theoretically unstable region. These discrepancies are believed to be due to a strong heat conduction in the walls. Koster and Müller [8] mention new experiments for nearly insulating walls which tend to confirm the present theoretical results.

CONCLUSIONS

The effects of hydrodynamic dispersion on free convection in a vertical Hele–Shaw cell have been studied. The heat transport is reduced due to dispersion. The stability domain in the α, Ra plane is slightly influenced by dispersion. The dispersion effects are small, but qualitatively different from the corresponding effects on convection in isotropic porous media [1].

The present dispersion effects will only be significant when d/h is comparable with unity. We will mention some effects which may then be more important than dispersion:

(1) Three-dimensional flow effects, leading to departure from the Hele–Shaw approximation.

(2) In an experiment there will be no slip at the upper and lower boundaries. This will be important within a distance of order d from the boundaries, but is not accounted for in the usual Hele–Shaw approximation. See the Appendix.

Future theories on Hele–Shaw cells should take these effects into account rather than dispersion.

REFERENCES

1. O. Kvernvold and P. A. Tyvand, Dispersion effects on thermal convection in porous media, *J. Fluid Mech.* **99**, 673–686 (1980).
2. R. A. Wooding, Instability of a viscous liquid of variable density in a vertical Hele–Shaw cell, *J. Fluid Mech.* **7**, 501–515 (1960).
3. R. N. Horne and M. J. O'Sullivan, Oscillatory convection in a porous medium heated from below, *J. Fluid Mech.* **66**, 339–352 (1974).
4. B. K. Hartline and C. R. B. Lister, Thermal convection in a Hele–Shaw cell, *J. Fluid Mech.* **79**, 379–391 (1977).
5. O. Kvernvold, On the stability of non-linear convection in a Hele–Shaw cell, *Int. J. Heat Mass Transfer* **22**, 395–400 (1979).
6. M. Poreh, The dispersivity tensor in isotropic and axisymmetric mediums. *J. Geophys. Res.* **70**, 3909–3913 (1965).
7. H. Neischloss and G. Dagan, Convective currents in a porous layer heated from below: The influence of hydrodynamic dispersion, *Physics Fluids* **18**, 757–761 (1975).
8. J. N. Koster and U. Müller, Private communication. Unpublished experiments performed at Kernforschungszentrum, Karlsruhe.

APPENDIX

ON THE INFLUENCE OF NO-SLIP CONDITION AT THE UPPER AND LOWER BOUNDARY

The Hele–Shaw approximation reduces the order of the equation of motion (1), so that restrictions on the tangential velocity at the boundaries must be abandoned. However, by introducing boundary layers in the model, it is possible to incorporate rigid boundaries. To study the effect of the boundary layers one must go back to the Navier–Stokes equation of motion. Due to the symmetry in the problem, it is sufficient to study the lower boundary layer.

In free convection, the vertical buoyancy creates pressure gradients which can in turn drive the flow horizontally. If d/h is not too large, the flow is largely horizontal in the boundary layers. As a first approximation, this horizontal flow is governed by a balance between the local pressure gradient $-G$ and viscous forces. The velocity distribution in the boundary layer is then given by the following boundary value problem (in dimensional quantities)

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{G}{\mu} \tag{A1}$$

$$\left.\begin{aligned} u = 0 \quad &\text{at} \quad y = \pm d/2 \\ u = 0 \quad &\text{at} \quad z = 0. \end{aligned}\right\} \tag{A2}$$

It has the exact analytical solution

$$u = \frac{G}{\mu}\left[\frac{1}{2}\left(\frac{d^2}{4} - y^2\right) - 4\frac{d^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} \times \cos\frac{(2n+1)\pi}{d}y \exp - \frac{(2n+1)\pi}{d}z\right]. \tag{A3}$$

From this velocity distribution one easily calculates the displacement thickness of the boundary layer

$$\Delta h = \frac{93}{\pi^5}d\sum_{n=1}^{\infty} n^{-5} \approx 0.3151d. \tag{A4}$$

These results may form the basis for future approximations of the no-slip effects. Generally, rigid boundaries are more important than dispersion. Both effects increase with the ratio d/h.

EFFETS DE LA DISPERSION SUR LA CONVECTION THERMIQUE DANS UNE CELLULE HELE–SHAW

Résumé—L'influence de la dispersion hydrodynamique sur la convection thermique dans une cellule Hele–Shaw verticale est étudiée théoriquement. On s'intéresse au mouvement permanent et supercritique des rouleaux, au transport thermique et à la stabilité du mouvement. Les effets de la dispersion sont trouvés faibles mais quantitativement différents des effets correspondants dans le cas de la convection en milieu poreux.

DISPERSIONSEINFLÜSSE AUF THERMISCHE KONVEKTION IN EINER HELE–SHAW–ZELLE

Zusammenfassung—Es wird der Einfluß der hydrodynamischen Dispersion auf die thermische Konvektion in einer senkrechten Hele–Shaw–Zelle theoretisch untersucht. Es werden die überkritische stationäre Drehbewegung, der Wärmeübergang und die Stabilität der Bewegung behandelt. Dabei wird festgestellt, daß die Dispersionseffekte gering sind, sich aber qualitativ von entsprechenden Effekten bei Konvektion in porösen Medien unterscheiden.

ВЛИЯНИЕ ДИСПЕРСИИ НА ТЕПЛОВУЮ КОНВЕКЦИЮ В ЯЧЕЙКЕ ХЕЛЕ–ШОУ

Аннотация — Теоретически исследуется влияние гидродинамической дисперсии на тепловую конвекцию в вертикальной ячейке Хеле–Шоу. Рассматриваются закритическое установившееся валообразное движение, а также перенос тепла и устойчивость движения. Установлено, что эффекты дисперсии незначительны, однако они качественно отличаются от соответствующих эффектов, наблюдаемых при конвекции в порах.